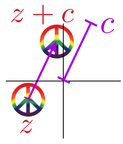
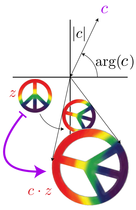
A brief story of the Fatou set

From one to two complex variables.

Complex numbers make it possible to add and multiply points in the plane. Thus, the arithmetic operations + and × become geometric transformations.

For a point of c in the plane, the transformation
$$z \mapsto z + c$$
translates every point z by c.





The transformation
$$z \mapsto c \times z$$
is the rotation of angle $\arg(c)$ composed with the scaling of factor $|c|$.

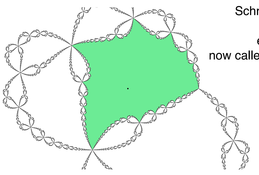
Hence given c, these operations enable us to define this plane transformation:
$$z \mapsto z \times z + c = z^2 + c$$

Then given an inital point z_o, we can define its orbit $z_0, z_1, z_2, \ldots, z_n, \ldots$ by:
$$z_1 = z_o^2 + c, z_2 = z_1^2 + c, \ldots, z_{n+1} = z_n^2 + c, \ldots$$

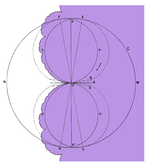
Some points have a stable behavior: nearby points follow similar trajectories. They form the Fatou set. Conversely, points with unstable behavior form the Julia set.

These dynamics can be generalized to arbitrary polynomials, and even to polynomials in two variables. In this case, the dynamics is defined on pairs of points in the plane, that is, on a four-dimensional space.

Julia and Fatou sets have fascinated mathematicians for over a century and remain an active area of research today.



1870 Schröder is the first to iterate planar transformations. He shows the existence of attracting domains, now called attracting Fatou components.



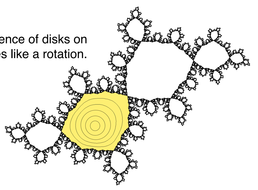
1897 Leau proves the Flower theorem: parabolic petals as Fatou components.

1906 Montel's Theorem implies that a point is stable if the iterates of some neighborhood avoid 3 points.

1917 Grand Prix de l'Académie des Sciences: Fatou and Julia lay the foundations of rational iteration theory.

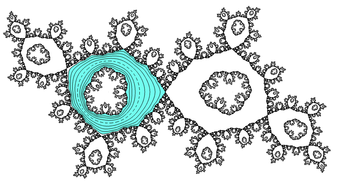
1918 Lattès gives an example of a rational function with empty Fatou set.

1942 Siegel proves the existence of disks on which the dynamics behaves like a rotation.



1983 Douady and Hubbard study the dynamics of quadratic polynomials and design the mathematical theory of the Mandelbrot set.

1984 Herman discovers his Fatou component: a ring on which the dynamics behaves like a rotation

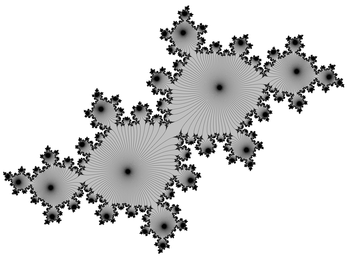


After simulation of a Herman ring by Fei Yong

1985 Sullivan shows that a Fatou component cannot be wandering: it must be be eventually periodic. Also it must be either attracting, parabolic or rotating.

1990 Yoccoz gives sharp conditions for the existence of Siegel disks.

2012 Buff and Chéritat prove the existence of Julia set with positive area



Arnaud Chéritat, approximation of a Julia set with positive area.

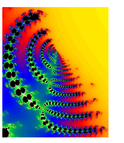
1986 Hubbard investigates the dynamics of the Hénon map:
$$(z, w) \mapsto (z^2 + a - b \cdot w, z)$$

1989 Friedland and Milnor isolate the Hénon maps
$$(z, w) \mapsto (P(z) - b \cdot w, z)$$
as the building block for all polynomial automorphisms.

1990 Milnor asks whether wandering Fatou domains exist for Hénon maps.

1991 Bedford and Smillie classify recurrent Fatou components for such automorphisms : they must be attracting or rotating.
Open question: do attracting Herman ring exist for dissipative maps?

1992 Fornaess and Sibony prove that the attracting Fatou components share the same boundary, so they have a very intricate topological structure.



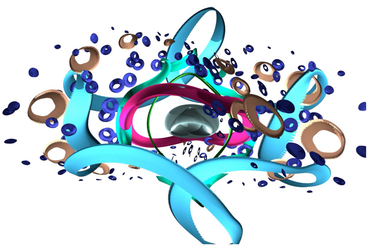
Hubbard's cutaway view of the Fatou set for a Hénon maps, 1990s.

1995 Hubbard and Oberste-Vorth propose a correspondence between the geometry of the Julia set of a Hénon map and that of its unstable slices.

2006 Lyubich and Peters complete the classification of periodic Fatou components for sufficiently dissipative maps.

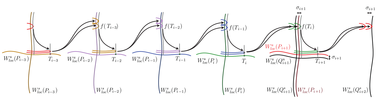
2016 Astorg, Buff, Dujardin, Peters and Raissy find two-dimensional polynomial mappings with wandering Fatou components.

2017 Computer pictures of conservative Fatou and Julia set by Ushiki suggest the existence of exotic rotation domains, that is Fatou components which are not balls but on which the dynamics acts as a rotation.



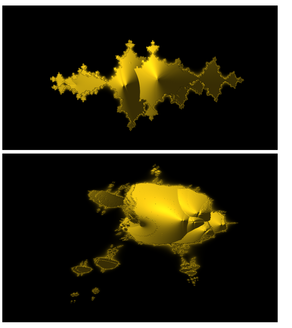
Ushiki simulation of exotic rotation domain for a Hénon map, 2021.

2023 Berger and Biebler show the existence of wandering Fatou components for Hénon maps. Moreover the behavior in such components is wild: no statistical description of the orbits is possible.



Construction of wandering Fatou component, after Berger-Biebler.

2026 Krikorian proves that exotic rotation domains and Herman rings exist.



First image of the filled Julia set for a Hénon map, Berger 2026